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LETTER TO THE EDITOR

Universality of the critical behaviour of the weakly disordered Baxter model

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Abstract. The two-point spin-spin correlation function of the weakly disordered Baxter model in the phase transition point is calculated. The corresponding critical exponent appears to be zero, i.e. the decay of the correlation function is slower than the power law: $R^{-\eta}$. Together with the previous result that the specific heat critical exponent of the weakly disordered Baxter model is also universal: $\alpha = 0$, it gives the complete set of universal critical exponents which describes the phase transition. The critical exponents of the weakly disordered Baxter model coincide with those of the weakly disordered Ising model.

In this letter the problem of phase transitions in weakly disordered (WD) systems will be considered. Due to intense theoretical and experimental studies it is now clear that a phase transition in WD spin systems is not 'smeared' but sharp, and is described by some universal critical exponents.

Let the disorder be described by a small parameter $\Delta \ll 1$, which could be e.g. the concentration of impurity bonds or mean value of the spin-spin coupling fluctuations. Then near the critical temperature in the narrow temperature range $\tau(\Delta) \sim \Delta^{1/\alpha_0}$, where α_0 is the specific heat exponent of a pure system, the critical behaviour is described by some universal critical exponents which could differ from those of the pure system (Harris and Lubensky 1974, Khmelnitskii 1975, Lubensky 1975).

The so-called Harris criterion is that the crossover to the new critical behaviour is to be expected only if the critical exponent $\alpha_0 > 0$, then a new specific heat critical exponent α_{imp} should be negative (Grinstein and Luther 1976). Otherwise when $\alpha_0 < 0$ WD is irrelevant for the critical behaviour (Harris 1974).

Recently, in accordance with this general statement, Newman and Riedel (1982) computed theoretically, and Birgeneau *et al* (1983) confirmed experimentally, that for the WD three-dimensional Ising model $\alpha_{\text{imp}} \simeq -0.09$ (cf $\alpha_0 \simeq +0.11$).

Two-dimensional (2D) weakly disordered systems are of special interest. The critical behaviour of the WD 2D Ising model (IM) was found exactly by Dotsenko and Dotsenko (1983). The critical exponents were shown to be

$$\begin{aligned}
&\alpha_{\text{imp}} = 0, \quad \nu_{\text{imp}} = 1, \quad \beta_{\text{imp}} = 0, \quad \gamma_{\text{imp}} = 2 \\
&\eta_{\text{imp}} = 0, \quad \mu_{\text{imp}} = \tfrac{1}{2}, \quad 1/\delta_{\text{imp}} = 0 \\
&(\text{cf } \alpha_0 = 0, \quad \nu_0 = 1, \quad \beta_0 = \tfrac{1}{8}, \quad \gamma_0 = \tfrac{7}{4} \\
&\eta_0 = \tfrac{1}{4}, \quad \mu_0 = \tfrac{8}{15}, \quad 1/\delta_0 = \tfrac{1}{15}).
\end{aligned} \tag{1}$$

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Although it is possible to study this system experimentally (see e.g. Als-Nielsen *et al* 1976) the main difficulty is due to an extremely small crossover temperature range $\tau(\Delta) \sim \exp(-\text{constant}/\Delta)$ and correspondingly a large crossover spatial scale $R(\Delta) \sim \tau(\Delta)^{-1}$. Note that although $\alpha_0 = \alpha_{\text{imp}} = 0$ the critical behaviour of the specific heat of the pure and WD 2D IMs are different: $C_0(\tau) \sim \ln(1/|\tau|)$, whereas $C_{\text{imp}}(\tau) \sim \ln\ln(1/|\tau|)$, where $\tau = 1 - T/T_c$.

Of course it is extremely difficult to check experimentally or by computer simulation the difference between ln and ln ln, but perhaps it is easier to detect the difference in exponents e.g. $\eta_0 = \frac{1}{4}$ and $\eta_{\text{imp}} = 0$, or $\gamma_0 = \frac{7}{4}$ and $\gamma_{\text{imp}} = 2$.

In a recent computer simulation by McMillan (1984) the value $\eta_{\text{imp}} = 0.16$ was obtained, which is noticeably lower than $\eta_0 = 0.25$. It seems that the non-zero value of η_{imp} is obtained due to the size of the system not being too large in comparison with the crossover scale.

The 2D IM, however, cannot answer the question of in which cases the WD is relevant for the critical behaviour since the specific heat exponent of this system $\alpha_0 = 0$.

The Baxter model (BM) is an interesting object of these studies. This model, solved exactly by Baxter (1971), can be considered as two 2D IMs coupled by four-spin interactions (see e.g. Baxter 1978). The strength of this coupling is described by some parameter g (the case $g = 0$ corresponds to two independent IMs). The critical exponents of the BM are continuously dependent on g, and the specific heat exponent, which is proportional to g for small coupling ($g \ll 1$), can be made both positive and negative.

It was shown by Dotsenko and Dotsenko (1983), that the specific heat critical exponent of the WD BM $\alpha_{\text{imp}} = 0$ (cf $\alpha_0 = 4g/\pi$) irrespectively of g. The specific heat critical behaviour proved to be logarithmic. For $g > 0$, when the specific heat of the pure model is divergent ($C_0(\tau) \sim |\tau|^{-4g/\pi}$), the specific heat of the WD model is still divergent, but logarithmically, $C_{\text{imp}}(\tau) \sim \ln\ln(1/|\tau|)$. On the other hand, for $g < 0$ ($C_0(\tau) \sim -|\tau|^{4|g|/\pi}$ is finite), the specific heat of the WD model does not remain the same as one would expect from the Harris criterion, but changes to a stronger cusp singularity: $C_{\text{imp}}(\tau) \sim [\ln\ln(1/|\tau|)]^{-1}$. Therefore the Harris criterion is not valid for the BM.

Recently Matthews-Morgan *et al* (1984) partially confirmed by computer simulation, the result mentioned above. For $g > 0$ the renormalisation group trajectories came into the IM fixed point $g = 0$. On the other hand for $g < 0$ the renormalisation yields some non-zero charge g_{ren}, $|g_{\text{ren}}| < |g|$. As was shown by Dotsenko and Feigelman (1981) and Dotsenko and Dotsenko (1983) for a special type of disorder this can actually be the case. But for the general type of disorder (spin–spin couplings fluctuate independently at each lattice bond) the fixed point should be $g_{\text{ren}} = 0$ for $g < 0$ as well, although the renormalisation trajectories could approach the fixed point more slowly.

The aim of this letter is to show that the two-point spin–spin correlation function of the WD BM is

$$\langle\overline{\sigma(0)\sigma(R)}\rangle \sim \exp[-(1/4\pi g)(\ln\ln R)^2], \tag{2}$$

that is, the critical exponent $\eta_{\text{imp}} = 0$. The result (2) coincides with that of the WD IM (Dotsenko and Dotsenko 1983).

Since the scaling limit of the BM near the critical point is described by the 2D fermion (Thirring) model with four-fermion interaction (Luther and Peshel 1975, Luther 1976) the renormalisation group methods similar to those used for the WD IM can also be applied to this case. We give here the schematic derivation of the result (2).

In the representation of two coupled Ising lattices, the Baxter model is described by the following classical energy (see e.g. Baxter 1978):

$$H_0 = -\sum_{xx'} J_{xx'}\sigma_x\sigma_{x'} - \sum_{yy'} J_{yy'}\mu_y\mu_{y'} - J_4 \sum_{\substack{xx'\\yy'}} \sigma_x\sigma_{x'}\mu_x\mu_{x'}. \tag{3}$$

Here $\sigma, \mu = \pm 1$ are Ising variables. The first two terms correspond to two IMs and the summation is performed over nearest neighbours on two different interpenetrating square lattices. The third term is responsible for the coupling of the two models.

In the scaling limit near the critical point the homogeneous BM ($J_{xx'} = J_{yy'} = J$) can equivalently be described by one complex (Dirac) fermion field or two real (Majorana) fermion fields (each one having two spinor components) ψ and χ with the Euclidean action

$$A = \int d^2x(-\tfrac{1}{2}\bar{\psi}\,\hat{\partial}\psi - \tfrac{1}{2}\bar{\chi}\,\hat{\partial}\chi - \tfrac{1}{2}m_0\bar{\psi}\psi - \tfrac{1}{2}m_0\bar{\chi}\chi + g_0(\bar{\psi}\psi)(\bar{\chi}\chi)) \tag{4}$$

where $\bar{\psi} = \psi^T\hat{\gamma}_5$, $\bar{\chi} = \chi^T\hat{\gamma}_5$; $m_0 \sim \tau \equiv (1 - T/T_c)$; $g_0 \simeq 2J_4$ for $J_4 \ll 1$ (see Luther and Peschel 1975). Quenched fluctuations of $J_{xx'}$ and $J_{yy'}$ can be described, in the scaling limit, by quenched Gaussian fluctuations of ψ and χ masses in the action (4) (see Dotsenko and Dotsenko 1983)

$$A = \int d^2x\left(-\frac{1}{2}\bar{\psi}\,\hat{\partial}\psi - \frac{1}{2}\bar{x}\,\hat{\partial}\chi - \frac{m_0 + \delta m_1(x)}{2}\bar{\psi}\psi - \frac{m_0 + \delta m_2(x)}{2}\bar{\chi}\chi + g_0(\bar{\psi}\psi)(\bar{\chi}\chi)\right). \tag{5}$$

Here

$$\overline{\delta m_i(x)\,\delta m_j(x)} = 4\Delta_0\,\delta_{ij}\,\delta(x - x'). \tag{6}$$

The parameter $\Delta_0 \sim (\overline{J^2} - (\bar{J})^2)/(\bar{J})^2$ describes the quenched bond fluctuations and is assumed to be small.

After averaging the free energy over the disorder, the effective theory is described by replicated action

$$H_0 = \int d^2x\left[\sum_{a=1}^{N}\left(-\frac{1}{2}\bar{\psi}^a\,\hat{\partial}\psi^a - \frac{1}{2}\bar{\chi}^a\,\hat{\partial}\chi^a - \frac{m_0}{2}\bar{\psi}^a\psi^a - \frac{m_0}{2}\bar{\chi}^a\chi^a + g_0(\bar{\psi}^a\psi^a)(\bar{\chi}^a\chi^a)\right)\right.$$
$$\left. + \frac{\Delta_0}{2}\sum_{a,b=1}^{N}[(\bar{\psi}^a\psi^a)(\bar{\psi}^b\psi^b) + (\bar{\chi}^a\chi^a)(\bar{\chi}^b\chi^b)]\right]. \tag{7}$$

In the final result one should put $N = 0$. The theory (7) can be studied by renormalisation group methods and in the course of renormalisation an additional vertex appears

$$\varkappa \sum_{a,b=1}^{N} (\bar{\psi}^a\psi^a)(\bar{\chi}^b\chi^b). \tag{8}$$

The renormalisation group equations for g, Δ and $\varkappa$ are

$$\begin{aligned} dg/d\xi &= -(4/\pi)g\Delta \\ d\Delta/d\xi &= -(4/\pi)\Delta^2 + (4/\pi)g\varkappa \\ d\varkappa/d\xi &= (4/\pi)g\Delta - (4/\pi)\varkappa\Delta \end{aligned} \tag{9}$$

where ξ is a renormalisation parameter. The asymptotic solutions (for $\xi \to \infty$) of these equations are

$$g(\xi) \sim \frac{|g_0|}{g_0} \frac{\pi}{4\xi \ln(4g_0\xi/\pi)} \left(1 - \frac{\ln \ln \xi}{\ln(4g_0\xi/\pi)} + \frac{1}{\ln(4g_0\xi/\pi)}\right)$$

$$\Delta(\xi) \sim \frac{\pi}{4\xi}\left(1 + \frac{1}{\ln(4g_0\xi/\pi)}\right), \qquad \varkappa(\xi) = g(\xi) \ln(g_0/g(\xi)) \tag{10}$$

(Dotsenko and Dotsenko 1984).

In the scaling limit the spin-spin correlation function $\langle\overline{\sigma(0)\sigma(R)}\rangle$ in the phase transition point can be represented as

$$\left\langle \exp\left(-\int_0^R \mathrm{d}x(\bar{\psi}^a T^{ab} \psi^b)\right)\right\rangle_H \tag{11}$$

where the averaging is performed over the action (7), (8) with $m_0 = 0$, and the $N \times N$ matrix $\hat{T}$ is

$$T = \begin{pmatrix} 2 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \tag{12}$$

(Dotsenko and Dotsenko 1983). In the course of renormalisation the term proportional to $(\bar{\chi}\chi)$ will appear in (11) and all diagonal components of the matrix $\hat{T}$ (12) will become non-zero. Therefore calculations should be performed for the expression

$$\langle\overline{\sigma(0)\sigma(R)}\rangle \sim \left\langle \exp\left(\int_0^R \mathrm{d}x \{[(\bar{\psi}^a\psi^b) + (\bar{\chi}^a\chi^b)] T_+^{ab} + [(\bar{\psi}^a\psi^b) - (\bar{\chi}^a\chi^b)] T_-^{ab}\}\right)\right\rangle_H \tag{13}$$

where

$$\hat{T}_\pm = \begin{pmatrix} t_1^\pm & 0 & 0 & \dots & 0 \\ 0 & t_2^\pm & 0 & \dots & 0 \\ 0 & 0 & t_2^\pm & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & t_2^\pm \end{pmatrix} \tag{14}$$

The renormalisation group equations for $\hat{T}_\pm$ can be easily derived

$$\mathrm{d}T_\pm^{ab}/\mathrm{d}\xi = -(2/\pi)\Delta(T_\pm^{ab} - (\mathrm{Tr}\,\hat{T}_\pm)\delta^{ab}) \pm (2/\pi) g T_\pm^{aa}\,\delta^{ab} \pm \varkappa(\mathrm{Tr}\,\hat{T})\,\delta^{ab} \tag{15}$$

where $\mathrm{Tr}\,\hat{T}_\pm = t_1^\pm + (N-1)t_2^\pm \equiv t_1^\pm - t_2^\pm$. Together with solutions (10), equations (15) give

$$t_1^{(+)2} - t_2^{(+)2} = \begin{cases} \dfrac{2\pi}{g_0\xi}(\ln((4g_0/\pi)\xi) + \ln\ln\xi), & g_0 > 0 \\[2ex] \dfrac{2\pi}{g_0\xi \ln^2((4g_0/\pi)\xi)}, & g_0 < 0 \end{cases} \tag{16}$$

$$t_1^{(-)2} - t_2^{(-)2} \sim \begin{cases} \dfrac{2\pi}{g_0\xi \ln^2((4g_0/\pi)\xi)}, & g_0 > 0 \\[2ex] \dfrac{2\pi}{g_0\xi}(\ln((4g_0/\pi)\xi) + \ln\ln\xi), & g_0 < 0. \end{cases} \tag{17}$$

Expanding (13) we get the multiple loop series with renormalised matrix elements $t_1^{\pm}(\xi)$ and $t_2^{\pm}(\xi)$. In particular the second-order loop is given by (for $N=0$)

$$-\frac{1}{4}\int_0^R dx_1\, dx_2 (T_+^{ab}(x_1-x_2)T_+^{ba}(x_2-x_1)+T_-^{ab}(x_1-x_2)T_-^{ba}(x_2-x_1))\frac{1}{(2\pi)^2(x_1-x_2)^2}$$

$$=-\frac{1}{2}\int_0^R dx_1\, dx_2[(t_1^{(+)2}-t_2^{(+)2})+(t_1^{(-)2}-t_2^{(-)2})]\frac{1}{(2\pi)^2(x_1-x_2)^2}. \quad (18)$$

From (16) and (17) one gets ($\xi\sim\ln|x_1-x_2|$)

$$t_1^{(+)2}-t_2^{(+)2}+t_1^{(-)2}-t_2^{(-)2}\sim\frac{2\pi}{|g_0|}\frac{\ln[(g_0/\pi)\ln(x_1-x_2)]+\ln\ln\ln(x_1-x_2)}{\ln(x_1-x_2)}. \quad (19)$$

It could be checked that the decreasing functions make higher-order loops convergent, so the main logarithmically divergent contribution for $R\to\infty$ comes from the second-order loop only. So in the main order one finds

$$\langle\overline{\sigma(0)\sigma(R)}\rangle\sim\exp\left(-\frac{1}{4\pi|g_0|}\int_0^R dx_1\, dx_2\frac{\ln\ln(x_1-x_2)}{(x_1-x_2)^2\ln(x_1-x_2)}\right) \quad (20)$$

which gives the result (2). The asymptotic behaviour (2) is valid beyond the crossover scale

$$R(\Delta)\sim\begin{cases}(\Delta_0/|g_0|)^{-\pi/4|g_0|}, & \Delta_0\ll|g_0|\\ \exp[(\pi/4|g_0|)\exp(\Delta_0/|g_0|)], & \Delta_0\gtrsim|g_0|.\end{cases} \quad (21)$$

Note finally that the result $\eta_{\text{imp}}=0$ implies that all other critical exponents of the WD BM are the same as in the WD IM and are given by (1). Therefore we come to the conclusion that the inhomogeneity make the critical behaviour of Ising-like two-dimensional systems 'more universal' than that of the homogeneous ones. The urgent problem now is to investigate whether this universal critical behaviour (equations (1)) is valid for any other two-dimensional weakly disordered systems.

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